Introduction to Game Theory

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1 Introduction to Game Theory

What is Game Theory? Game theory is a mathematical framework for analyzing situations in which the outcomes depend on the interactions of multiple decision-makers (players). It studies the strategic decisions where the success of one player's strategy depends on the strategies adopted by others. Game theory is widely applicable in economics, political science, biology, and many other disciplines.

There are various types of games analyzed in game theory, each with its own unique characteristics and applications:

- **Normal-Form (or Strategic-Form) Games**: Represented by a matrix showing the payoffs for each strategy combination chosen simultaneously by the players.
- **Extensive-Form Games**: Depicted as a decision tree showing the sequential nature of the game, with decision nodes representing points where players make choices.
- **Repeated Games**: Games played over several periods, where players can adjust their strategies based on previous outcomes.
- **Bayesian Games**: Games where players have incomplete information about other players, such as their payoffs or types, and must form beliefs and strategies based on this uncertainty.

2 Normal-Form Games

The normal-form representation is the most straightforward and concise method for depicting a game. Normal-Form games are also known as **strategic form games or matrix games**.

Definition 2.1 (Normal-Form Game). *A* ***normal-form game*** *is a tuple* $G = (I, \{A_i\}_{i \in I}, \{u_i\}_{i \in I})$ *such that:*

1. *$I = \{1, 2, \ldots, N\}$ is the set of players.*

2. *A_i is the set of actions available to player i.*

3. *$u_i : A_1 \times \cdots \times A_N \to \mathbb{R}$ is the payoff function for player i.*

*We write $A = \prod_{i=1}^{N} A_i$ for the set of **action profiles**, $a = (a_1, \ldots, a_n) \in A$ and $A_{-i} = \prod_{j \neq i}^{N} A_j$ for the actions of all players but i.*

Definition 2.2. *In a normal-form game G, let $\Sigma_i = \Delta(A_i)$ denote the set of probability distributions over the set A_i and call an element $\sigma_i \in \Sigma_i$ a (mixed) strategy of player i. We say that a strategy σ_i is:*

1. *A **pure strategy** if it assigns probability 1 to a single action. We will denote by a_i a pure strategy σ_i that assigns probability 1 to action a_i.*

2. *A **(mixed) strategy** if it assigns probabilities to the actions in A_i.*

3. *A **strictly mixed strategy** if it assigns strictly positive probabilities on more than one action.*

We denote by $\Sigma = \prod_{i=1}^{N} \Sigma_i$ the set of strategies profiles $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma$ and $\Sigma_{-i} = \prod_{j \neq i}^{N} \Sigma_j$ for the strategies of all players but i.,

Note that in this form each player chooses, individually and independently, an action $a_i \in A_i$ and then the action profile is formed $a \in A$. No player knows the choice of the other player before choosing its own action.
Note that by independence, the expected payoff of a player i sum of payoffs for all possible action profiles weighted by the probability of each action profile occurring, i.e.:

$$\mathbb{E}[u_i(\sigma)] = \sum_{a \in A} u_i(a) \cdot \prod_{j \in I} \sigma_j(a_j).$$

What does it mean to solve a game? It involves predicting how the game will unfold, identifying the strategies players will choose, and determining the resulting outcome of the game.

2.1 Solution concepts:

We will explore two basic assumptions that derive different solutions concepts and impose restrictions on the way players can play a game: **Rationality and Common Knowledge of Rationality**. We will ask two primary questions and derive different solutions concepts for each:

1. **What strategies will never be played?**

- Assumption of Rationality gives Dominant strategy equilibrium.
- Assumptions of Rationality and Common Knowledge of Rationality give Iterated elimination of strictly dominated strategies equilibrium (IESDS or ISD).

2. **What strategies could potentially be played?**

   - Assumptions of Rationality and Common Knowledge of Rationality give the concept of Rationalizability.

Finally, we will compare these solution concepts.

3 Strict Dominance

In this section, we assume **rationality**. That is, we assume that players are rational: they never play bad strategies or that players maximize their expected payoffs.

Definition 3.1 (Strictly Dominated strategy). *In a normal-form game G, a pure strategy $a_i \in \Sigma_i$ is strictly dominated if there exists $\sigma'_i \in \Sigma_i$ such that*

$$u_i(\sigma'_i, a_{-i}) > u_i(a_i, a_{-i}), \quad \forall a_{-i} \in \Sigma_{-i}.$$

Note that since mixed strategies are convex combinations of pure strategies, we have,

$$u_i(\sigma'_i, \sigma_{-i}) > u_i(a_i, \sigma_{-i}), \quad \forall \sigma_{-i} \in \Sigma_{-i}.$$

We say that $\sigma_i \in \Sigma_i$ is strictly dominated if there exists $\sigma'_i \in \Sigma_i$ that strictly dominates σ_i.

Remark: A rational player will never play a strictly dominated strategy.

Example 3.1. *Consider the Prisoners' dilemma below:*

	C	L
C	$1, 1$	$-1, 2$
L	$2, -1$	$0, 0$

For each player, C is strictly dominated by L. Since neither player plays C, we conclude that strategy (L, L) will be played.

Remark: In this example, rationality alone is sufficient to predict the chosen strategy, as choosing lying (L) is the rational course of action since playing C is worst than playing L, **regardless** of what the opponent does.

However, there are two important things to consider:

Example 3.2 (In many games, rationality alone will not give a unique prediction).

	L	M	H
L	$6, 6$	$-2, 8$	$0, 4$
M	$8, 2$	$4, 4$	$1, 3$
H	$4, 0$	$3, 1$	$2, 2$

Strategy L is never played by either player, it is a dominated strategy. Neither M nor H is dominated, thus, rationality alone does not offer a unique solution.

Example 3.3 (Verification of mixed strategies). *Note to show that a strategy is not strictly dominated, it is not enough to verify that it is not dominated by only pure strategies, we also need to prove that there exists no mixed strategy that strictly dominates it.*

	L	R
U	$3, 0$	$-1, 0$
M	$0, 0$	$0, 0$
D	$-1, 0$	$3, 0$

No pure strategy is strictly dominated by any pure strategy. However, M is indeed strictly dominated by the mixed strategy $\left(\frac{1}{2}U, \frac{1}{2}D\right)$.

3.1 Dominant Strategy Equilibrium

A dominant strategy is always the best thing you can do, regardless of what your opponents choose.

Definition 3.2 (Strictly dominant strategy). *A strategy $\sigma_i \in \Sigma_i$ is a strictly dominant strategy for player i if every other strategy of i is strictly dominated by it. That is,*

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}) \quad \textit{for all } \sigma_i' \in \Sigma_i, \sigma_i' \neq \sigma_i, \textit{ and all } \sigma_{-i} \in \Sigma_{-i}.$$

With this, we can define the first solution concept:

Definition 3.3 (Strict dominant strategy equilibrium). *The strategy profile $\sigma^D \in \Sigma$ is a strict dominant strategy equilibrium if $\sigma_i^D \in \Sigma_i$ is a strict dominant strategy for all $i \in \mathcal{I}$.*

If we can find a dominant strategy equilibrium for other games, then this solution concept has a very appealing property:

Proposition 3.1. *If a Normal-form game G has a strictly dominant strategy equilibrium σ^D, then σ^D is the unique dominant strategy equilibrium.*

What if no player has a dominated strategy, which implies that no player has a dominant strategy either:

Example 3.4 (Battle of the sexes). *A game with no equilibrium under the solution concept of strict dominance.*

	B	F
B	$3,2$	$0,0$
F	$0,0$	$2,3$

Therefore, if we adhere strictly to the concept of strict dominance, we will find that many games lack an equilibrium.

4 Iterated Elimination of Strictly Dominated Strategies (IESDS)

Recall that rationality alone implies to things:

1. A rational player will never play a dominated strategy.
2. If a rational player has a strictly dominant strategy then he will play it.

The second point gives the solution concept of strict dominance. However, we need a solution concept that is more general.

Starting from the assumption of Rationality and its implication that players will never play a dominated strategy, we ruled out what players **will not do**.

We now impose an additional assumption on the structure of the game and that rationality of each player is common to all players, also know as **Common Knowledge**. That is, if all players understand that no one will choose a strictly dominated strategy, they can effectively disregard those strategies, enabling us to do much more than merely identifying strategies that rational players will avoid. We can now ignore strictly dominated strategies, giving rise to smaller, restrictive games with fewer strategies.

Example 4.1. *A normal-form game where the iterative deletion of strictly dominated strategies results in a single remaining pure strategy.*

	L	M	R
U	$3,2$	$1,1$	$4,0$
D	$1,2$	$4,1$	$3,5$

$\underset{(1)}{\Rightarrow}$

	L	R
U	$2,2$	$4,0$
D	$1,2$	$3,5$

$\underset{(2)}{\Rightarrow}$

	L	R
U	$2,2$	$4,0$

$\underset{(3)}{\Rightarrow}$

	L
U	$2,2$

(1) None of Player's 1 (P1) strategies is dominated. But, player 2 (P2) will never choose M (*because of rationality*). (2) After deleting strategy M, P1 will never choose D, as it is strictly dominated (*because P1 knows that P2 is rational*). (3) Strategy R is dominated for P2 (*because P2 knows that P1 knows that P2 is rational*). Thus, only the pure strategy (U,L) survives **iterated deletion of strictly dominated strategies**.

We now formalize this procedure to derive a new solution concept for the strategy σ that survives iterated deletion of strictly dominated strategies.

Definition 4.1. *In a normal-form game G, for each $i \in I$ and each $k \in \mathbb{N}$, we define $S_i^0 = A_i$ and*

$$S_i^k = S_i^{k-1} \setminus \left\{ a_i \in S_i^{k-1} \mid \exists \sigma_i \in \Delta(S_i^{k-1}), \forall a_{-i} \in S_{-i}^{k-1}, u_i(\sigma_i, a_{-i}) > u_i(a_i, a_{-i}) \right\}.$$

and the set of player i's pure strategies that survive iterated deletion of strictly dominated strategies by

$$D_i^\infty = \bigcap_{k=0}^{\infty} S_i^k.$$

And we can extend it to mixed strategies:

Definition 4.2. *In a normal-form game G, for each $i \in I$, we define the set of player i's mixed strategies that survive iterated deletion of strictly dominated strategies by*

$$\Sigma_i^\infty = \Delta(D_i^\infty) \setminus \left\{ \sigma_i \in \Delta(D_i^\infty) : \exists \sigma_i' \in \Delta(D_i^\infty), \forall a_{-i} \in D_{-i}^\infty, u_i(\sigma_i', a_{-i}) > u_i(\sigma_i, a_{-i}) \right\}.$$

Example 4.2. *A normal-form game with $\Sigma_i^\infty \neq \Delta(D_i^\infty)$.*

	L	R
U	$1, 0$	$-2, 0$
M	$-2, 0$	$1, 0$
D	$0, 0$	$0, 0$

For each player, $D_i^\infty = S_1^0 = A_i$. Thus, $\left(\frac{1}{2}U, \frac{1}{2}M\right) \in \Delta(D_i^\infty)$, but, $\left(\frac{1}{2}U, \frac{1}{2}M\right)$ is strictly dominated by D. That is, $\left(\frac{1}{2}U, \frac{1}{2}M\right) \notin \Sigma_i^\infty$.

Theorem 4.1. *Suppose that either (1) each A_i is finite, or (2) each $u_i(\sigma_i, \sigma_{-i})$ is continuous and each A_i is compact (i.e., closed and bounded). Then $D_i^\infty \neq \emptyset$.*

Remarks:

- An infinite Normal-Form Game can yield $D_i^\infty = \emptyset$.
- In general $\Sigma_i^\infty \neq \Delta(D_i^\infty)$.

Contrary to the strict dominance solution concept, we can utilize ISD in any game, as it does not require the presence of strictly dominant or strictly dominated strategies. Nevertheless, the presence of strictly dominated strategies helps ISD illustrate how the assumption of common knowledge of rationality influences players' behavior. Without them, the process can fail to provide a unique solution (see Battle of the sexes example).

Proposition 4.1. *If for a game G, σ^* is a strict dominant strategy equilibrium, then σ^* uniquely survives Iterated Elimination of Strictly Dominated Strategies (IESDS or ISD).*

The above proposition tells us that whenever strict dominance results in a unique payoff, then ISD will result in the same unique payoff after one round.

Remarks:

- ISD is a more widely applicable solution concept.
- Game Theory relies a lot in the strong assumption of common knowledge of rationality.

5 Rationalizability

The two solutions concepts previously described are based on eliminating actions that players will never play. Now, we turn to answer what strategies **could potentially** be played under common knowledge of rationality.

Note that if a strategy is not strictly dominated for a player, then it suggest that **under some conditions** this strategy could be played! We want to **rationalize** the behavior for when a player would choose such strategy.

We can conceptualize it as follows: if a strategy σ_i is not strictly dominated, there must be some combinations of strategies from player i's opponents for which σ_i is the optimal choice for player i. We now turn to formalize this.

5.1 Correlated Rationalizability

Beliefs about Opponents' Strategies. In a normal-form game G, we call $\mu_{-i} \in \Delta(A_{-i})$ player i's belief about the opponents' strategies. Given a belief μ_{-i}, if player i plays a strategy $\sigma_i \in \Sigma_i$, then his expected payoff is:

$$u_i(\sigma_i, \mu_{-i}) = \sum_{a \in A} u_i(a)\sigma_i(a_i)\mu_{-i}(a_{-i}).$$

Remark: Recall $A_{-i} = \prod_{j \neq i} A_j$. It is important to note that $\Delta\left(\prod_{j \neq i} A_j\right) \neq \prod_{j \neq i} \Delta(A_j)$. While the left-hand side is the set of **correlated mixed strategies** of all players but i, the right-hand side is the set of **independent mixed strategies** of all players but i.

Definition 5.1 (Never-best response). *In a normal-form game G, a strategy $\sigma_i \in \Sigma_i$ is a never-best response if for each $\mu_{-i} \in \Delta(A_{-i})$, there exists $\sigma_i' \in \Sigma_i$ such that*

$$u_i(\sigma_i', \mu_{-i}) > u_i(\sigma_i, \mu_{-i}).$$

Given any belief a player holds about their opponents' behavior, they must choose an action that maximizes their expected payoff according to these beliefs.

Proposition 5.1. *If σ_i is a strictly dominated strategy for player i, then it is a never-best response.*

Definition 5.2. *In a normal-form game G, for each $i \in \mathcal{I}$ and each $k \in \mathbb{N}$, we define $\tilde{S}_i^0 = A_i$ and*

$$\tilde{S}_i^k = \tilde{S}_i^{k-1} \setminus \left\{ a_i \in \tilde{S}_i^{k-1} \mid \forall \mu_{-i} \in \Delta(\tilde{S}_{-i}^{k-1}), \exists \sigma_i' \in \Delta(\tilde{S}_i^{k-1}), u_i(\sigma_i', \mu_{-i}) > u_i(a_i, \mu_{-i}) \right\}.$$

pure strategies that are never-best responses *We define player i's set of correlated rationalizable pure strategies by*

$$CR_i^\infty = \bigcap_{k=0}^{\infty} \tilde{S}_i^k.$$

The logic behind this iterative process is as follows: At each step, it identifies which actions could rational players potentially take? Each player then infers that no rational player will ever choose pure strategies that are never best responses. Consequently, players will believe that such pure strategies will not be played with positive probabilities. This mutual understanding becomes common knowledge, thereby justifying the elimination of these pure strategies from consideration in the game.

Definition 5.3. *In a normal-form game G, we define the set of player i's (correlated) rationalizable mixed strategies by*

$$\tilde{\Sigma}_i^\infty = \Delta(\tilde{RC}_i^\infty) \setminus \left\{ \sigma_i \in \Delta(\tilde{RC}_i^\infty) \mid \forall \mu_{-i} \in \Delta(\tilde{RC}_{-i}^\infty), \exists \sigma_i' \in \Delta(\tilde{RC}_i^\infty), u_i(\sigma_i', \mu_{-i}) > u_i(\sigma_i, \mu_{-i}) \right\}.$$

As with IESDS, we have that generally, $\Sigma_i^\infty \neq \Delta(\tilde{S}_i^\infty)$, but this is not true in general.

Now, it is super important to note that the concept of never-best response is defined over $\Delta(A_{-i})$ for the following results to hold.

Lemma 5.1. *In a finite normal-form game G, a strategy $\sigma_i \in \Sigma_i$ is a never-best response if and only if it is strictly dominated.*

Proof. If part is inmediate. Only if part relies on proving the contrapositive using the Separating Hyperplane Theorem (Taken from [2]).

If σ_i is not strictly dominated then it is not a never-best response. Suppose that σ_i is not strictly dominated. That is, there exists no $\sigma_i' \in \Sigma_i$ such that for each $a_{-i} \in A_{-i}$,

$$u_i(\sigma_i', a_{-i}) > u_i(\sigma_i, a_{-i}).$$

There are, in total, $L = \prod_{j \neq i} |A_j|$ possible action profiles of all players but i, and we enumerate them by $a_{-i}^1, a_{-i}^2, \ldots, a_{-i}^L$. To invoke the Separating Hyperplane Theorem, we define a set Y by

$$Y = \left\{ \left(u_i(\sigma_i', a_{-i}^l) - u_i(\sigma_i, a_{-i}^l) \right) : \sigma_i' \in \Delta(A_i) \right\} \subseteq \mathbb{R}^L.$$

Since Y is a non-empty, convex set and $Y \cap \mathbb{R}_+^L = \emptyset$, it follows from the Separating Hyperplane Theorem that there exist some $c \in \mathbb{R}$ and $v \in \mathbb{R}^L \setminus \{0\}$ such that for each $x \in \mathbb{R}_{++}$ and each $y \in Y$,

$$v \cdot x \geq c > v \cdot y.$$

It is immediate that $v \in \mathbb{R}^L \setminus \{0\}$. Normalize v to define $\mu_{-i} = \frac{v}{\sum_l v_l} \in \Delta(A_{-i})$. Then, $\mu_{-i} \cdot x \geq \tilde{c} \geq \mu_{-i} \cdot y$ for each $y \in Y$, where $\tilde{c} = \frac{c}{\|v\|_1}$. Since $\mu_{-i} \cdot x \geq 0$ and $\tilde{c} \leq 0$, we have $0 \geq \mu_{-i} \cdot y$ for each $y \in Y$. That is, for each σ_i',

$$\sum_{l=1}^{L} u_i(\sigma_i', a_{-i}^l)\mu_{-i}(a_{-i}^l) \leq \sum_{l=1}^{L} u_i(\sigma_i, a_{-i}^l)\mu_{-i}(a_{-i}^l),$$

or equivalently $u_i(\sigma_i', \mu_{-i}) \leq u_i(\sigma_i, \mu_{-i})$. Thus, σ_i is not a never-best response. □

Proposition 5.2 (Correlated Rationalizability = Iterated Strict Dominance). *In a finite form game G, $D_i^\infty = RC_i^\infty$ and $\Sigma_i^\infty = \tilde{\Sigma}_i^\infty$.*

5.2 Independent Rationalizability

Now, we consider the case where instead of defining a never-best response over the set of correlated mixed strategies$\Delta\left(\prod_{j \neq i} A_j\right)$ we defined it over the set of independent mixed strategies of all players but i, $\prod_{j \neq i} \Delta(A_j)$. We formalize this as follows:

Definition 5.4. *Given a normal-form game G, let $\hat{S}_i^0 = A_i$ and for each $k \in \mathbb{N}$,*

$$\hat{S}_i^k = \hat{S}_i^{k-1} \setminus \left\{ a_i \in \hat{S}_i^{k-1} \mid \forall \mu_{-i} \in \prod_{j \neq i} \Delta(\hat{S}_j^{k-1}), \exists \sigma_i' \in \Delta(\hat{S}_i^{k-1}), u_i(\sigma_i', \mu_{-i}) > u_i(a_i, \mu_{-i}) \right\}.$$

We define player i's set of independent rationalizable pure strategies by

$$R_i^\infty = \bigcap_{k=0}^{\infty} \hat{S}_i^k.$$

Definition 5.5. *In a normal-form game G, we define the set of player i's independent rationalizable mixed strategies by*

$$\hat{\Sigma}_i^\infty = \Delta(R_i^\infty) \setminus \left\{ \sigma_i \in \Delta(R_i^\infty) \mid \forall \mu_{-i} \in \prod_{j \neq i} \Delta(\hat{S}_j^{k-1}), \exists \sigma_i' \in \Delta(R_i^\infty), u_i(\sigma_i', \mu_{-i}) > u_i(\sigma_i, \mu_{-i}) \right\}.$$

Remark: In general $R_i^\infty \subseteq RC_i^\infty = D_i^\infty$

- In two players game we have that $R_i^\infty = RC_i^\infty = D_i^\infty$.
- Generally, in games with more than two players $R_i^\infty \subseteq RC_i^\infty = D_i^\infty$.

When does the equivalence break down? In ISD, $\sigma_{-i} \in \Delta(S_{-i}^n)$, his means that it includes probability distributions achievable only through correlated actions by player i's opponents. In independent rationalizability, however, the focus is on the space of independent mixed strategies for player i's opponents, which forms a nonconvex set and $\prod_{j \neq i} \Delta(\hat{S}_j^n) \subseteq \Delta(S_{-i}^n)$.

Example 5.1 ($D_i^\infty \not\subset R_i^\infty$). *Consider the following three-player game in which all of the player's payoffs are the same.*

$$A_1 = \{A, B\}, \quad A_2 = \{C, D\}, \quad A_3 = \{M_1, M_2, M_3, M_4\}$$

	M_1		M_2		M_3		M_4	
	C	D	C	D	C	D	C	D
A	8	0	4	0	0	0	3	3
B	0	0	0	4	0	8	3	3

Claim: M_2 is not strictly dominated. This is easy to see by inspection. For M_2 to be strictly dominated, there needs to be another M_i (or mixed strategy) such that for every possible action of players 1 and 2, the outcomes in M_i are better than in M_2.

Claim: M_2 is never a best response to any mixed strategy of players 1 and 2.

Proof. suppose $\sigma_1(A) = p$ and $\sigma_2(C) = q$ and towards a contradiction suppose not.

$$u_3(M_2, p, q) = 4pq + 4(1-p)(1-q) = 8pq + 4 - 4p - 4q \geq \begin{cases} 8pq, \\ 8 + 8pq - 8(p+q), \\ 3 \end{cases}$$

From the first two equations we get:

$$\begin{aligned} 1 &\geq p + q, \\ 1 &\leq p + q, \end{aligned} \implies p + q = 1.$$

Substituting in the third equation: $pq \geq -1 + 4 = 3 \implies pq \geq \frac{3}{8}$, Thus, substituting in the inequality $1 \geq p + q$, we get:

$$p^2 + pq \leq p \implies p^2 - p + \frac{3}{8} \leq 0 \implies (p - \frac{1}{2})^2 + \left(q - \frac{1}{8}\right)^2 \leq 0$$
$$\implies (p - \frac{1}{2})^2 \leq -\frac{1}{8}, \quad \text{which has no real solutions.}$$

$$\therefore M_2 \text{ is never a best response.}$$

□

5.3 Relation between D_i^∞ and R_i^∞

To sum up,

- The set of strictly dominated strategies is a strict subset of the set of never-best response strategies, when defined over the space of independent mixed strategies for player i's opponents.
- Rationalizable strategies represent a further refinement of the strategies that survive iterated strict dominance.
- $R_i^\infty \subseteq D_i^\infty$.

6 Nash Equilibrium

This section relies heavily on the notes of [2]. The concept of Nash equilibrium (NE) is fundamental in game theory, representing a state where each player's strategy is the best response to the strategies of the others. This steady-state assumption implies that no player can unilaterally improve their payoff by deviating from their chosen strategy, given the strategies of the other players. That is, there is no profitable deviation for any player. Nash equilibrium provides a robust solution concept in strategic games by encapsulating mutual best responses. However, the presence of multiple Nash equilibria, without a clear ranking, can complicate predictions. One key feature of a Nash equilibrium is the common belief shared by all players regarding each other's actions, fostering consistency and rational expectations across the game. Note that a Nash equilibrium does not attempt to examine the process by which a steady state is reached.

Definition 6.1 (Nash Equilibrium). *In a normal-form game G, a strategy profile σ is a Nash equilibrium if for each $i \in I$ and each $\sigma_i' \in \Sigma_i$,*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}).$$

A Nash equilibrium is a pure-strategy Nash equilibrium if every player plays a pure strategy.

Proposition 6.1 (Property of NE). *In a normal-form game G, let σ^* be a Nash equilibrium. For all $a_i, a_i' \in A_i$ such that $\sigma_i^*(a_i), \sigma_i^*(a_i') > 0$, it must be that*

$$u_i(\sigma_i^*, \sigma_{-i}^*) = u_i(a_i, \sigma_{-i}^*) = u_i(a_i', \sigma_{-i}^*).$$

Proof. Assume, for contradiction, that there exist $a_i, a_i' \in A_i$ such that $\sigma_i^*(a_i) > 0$ and $\sigma_i^*(a_i') > 0$, but $u_i(a_i, \sigma_{-i}^*) > u_i(a_i', \sigma_{-i}^*)$. This implies that player i can achieve a higher payoff by increasing the probability of playing a_i instead of a_i'. □

6.1 Existence

To prove the existence of a Nash Equilibrium, it is convenient to use the following definition

Definition 6.2. *In a normal-form game G, player i's best-response correspondence is a correspondence $B_i : \Sigma_{-i} \to 2^{\Sigma_i}$ such that for each $\sigma_{-i} \in \Sigma_{-i}$, $B_i(\sigma_{-i})$ is the set of his optimal strategies given that (player i believes) players $-i$ play σ_{-i}. That is,*

$$B_i(\sigma_{-i}) = \arg\max_{\sigma_i' \in \Sigma_i} u_i(\sigma_i', \sigma_{-i}).$$

Now, with this, we can immediately redefine an equivalent definition of Nash equilibrium.

Definition 6.3. *In a normal-form game G, $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Sigma$ is a Nash equilibrium if for each $i \in I$, $\sigma_i \in B_i(\sigma_{-i})$.*

We will use Kakutani's Fixed Point Theorem. The following definitions are useful:

Definition 6.4. *Let $X, Y \subseteq \mathbb{R}^n$, with $n < \infty$. Let $F : X \to 2^Y$ be a correspondence.*

- *F is non-empty-valued if for each $x \in X$, $F(x)$ is non-empty*
- *F is convex-valued if for each $x \in X$, $F(x)$ is convex.*
- *F has a closed graph if the graph $Gr(F)$ is closed in $X \times Y$ (with respect to the relative topology), where we define $Gr(F)$ by*

$$Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

6.2 Nash Equilibrium and Iterated Strict Dominance/Rationalizability

Proposition 6.2. *In a finite normal-form game G, suppose that σ^* is a Nash equilibrium. For each $i \in I$, if $\sigma_i^*(a_i) > 0$ then:*

1. *a_i survives iterated deletion of strictly dominated strategies. That is, $a_i \in S_i^\infty$.*
2. *a_i is correlated rationalizable. That is, $a_i \in CR_i^\infty$.*

Theorem 6.1 (Kakutani's Fixed Point Theorem). *Let X be a non-empty, compact, convex subset of $\mathbb{R}^n$, with $n < \infty$. Let $F : X \to 2^Y$ be a non-empty-valued, convex-valued correspondence with a closed graph. Then, there exists some $x^* \in X$ such that $x^* \in F(x^*)$.*

Theorem 6.2 (Nash's Existence Theorem). *In a finite normal-form game G, there exists a Nash equilibrium.*

Proof. Let $B : \Sigma \to 2^\Sigma$ be defined by $B(\sigma) = (B_1(\sigma_{-1}), \ldots, B_N(\sigma_{-N}))$ for each $\sigma \in \Sigma$. To apply Kakutani's Fixed Point Theorem, we verify the following properties:

1. Σ_i is non-empty, compact, and convex.

2. B_i is non-empty-valued. To see this, note that u_i is continuous in σ_i and Σ_i is compact.

3. B_i is convex-valued. To see this, let σ_i, σ_i' be in $B_i(\sigma_{-i})$. By Proposition 1, all $a_i \in \text{supp}(\sigma_i)$ and all $a_i' \in \text{supp}(\sigma_i')$ yield the same payoff. Hence, player i is indifferent to any randomization over $\text{supp}(\sigma_i) \cup \text{supp}(\sigma_i')$.

4. B_i has a closed graph. To see this, let $(\sigma_k, \sigma_k') \in \text{Gr}(B)$ such that $\sigma_k' \to \sigma'$ and $\sigma_k \to \sigma$ as $k \to \infty$. It suffices to show that $\sigma_i \in B_i(\sigma_{-i})$ for each $i \in N$. Note that for each $k \in N$, $u_i(\sigma_k', \sigma_{-i}) \geq u_i(\sigma_{k_i}', \sigma_{-i})$ for each σ_i'. Hence, $\lim_{k\to\infty} u_i(\sigma_k', \sigma_{-i}) \geq \lim_{k\to\infty} u_i(\sigma_i', \sigma_{-i})$ for each σ_i'. Since u_i is continuous, $u_i(\sigma', \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for each σ_i'.

Hence, B satisfies all the conditions for Kakutani's Fixed Point Theorem. There thus exists some $\sigma^* \in \Sigma$ such that $\sigma^* \in B(\sigma^*)$.

□

Example 6.1 (Aumann's (1974) game). *Consider the following game*

	A	B
A	$5, 1$	$0, 0$
B	$4, 4$	$1, 5$

There are three Nash equilibria: two pure-strategy Nash equilibria (A, A) and (B, B); and one mixed-strategy Nash equilibrium $\frac{1}{2}(A, B)$ for each player.

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